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Convergence of Generalized Inverses and Spline Projectors

SAICHI IZUMINO

Faculty of Education, Toyama University, Toyama-shi 930, Japan

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INTRODUCTION

The theory of generalized inverses has provided operator theoretic approaches in many areas related to least-squares solutions of linear equations (see, e.g., [11]). As its recent applications (in approximation theory) we find, for example, Chang [4] on minimum norm interpolation, Delvos [5] on interpolating splines, Delvos and Shempp [6] on optimal approximation, and Groetsch [8] on generalized splines.

Our aim is to show that the generalized inverse method is applicable to the convergence problem of abstract spline projectors; we shall give some refinements of theorems due to de Boor [3] and Shekhtman [13] on the convergence of abstract splines.

1. PRELIMINARIES

Let H and K be Hilbert spaces, and let $A \in B(H, K)$ be a bounded linear operator from H into K. If A has closed range, then, as is well-known [2] [7], there exists a unique operator $A^{\dagger} \in B(K, H)$ satisfying the following four Penrose identities:

$$AA^{\dagger}A = A, \qquad A^{\dagger}AA^{\dagger} = A^{\dagger}, \qquad (AA^{\dagger})^* = AA^{\dagger}, \qquad \text{and} \qquad (A^{\dagger}A)^* = A^{\dagger}A.$$

The operator $A^{\dagger}$ is called the generalized inverse of A. We shall denote by $\mathrm{CR}(H, K)$ the set of all operators in $B(H, K)$ with closed range. If we write $\operatorname{ran} A$ and $\ker A$ for the range and the kernel of A ($\in \mathrm{CR}(H, K)$), respectively, then the products $AA^{\dagger}$ and $A^{\dagger}A$ are orthoprojectors (or orthogonal projections) onto $\operatorname{ran} A$ and $(\ker A)^{\perp}$, the orthocomplement of $\ker A$, respec-

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tively. As the variational properties of generalized inverses, we know [7, 9] that for each $v \in K$ the set of all least-squares solutions of the equation $Au = v$, i.e., the set of all minimizers of $\|Au - v\|$ is given by

$$\{A^{\dagger}v + (1 - A^{\dagger}A)w : w \in H\}, \tag{1.1}$$

and furthermore, $u_0 := A^{\dagger}v$ is the (unique) best approximate solution of $Au = v$, i.e., u_0 is the minimizer with minimal norm.

Next, in order to define a spline interpolant (or abstract spline), let $T \in \mathrm{CR}(H, K)$ and let L be a closed linear subspace of H. Then for a given $x \in H$, an element $y \in x + L^{\perp}$ is called a *(T, L)-spline interpolant* of x [3] if

$$\|Ty\| = \inf\{\|Tz\| : z \in x + L^{\perp}\}.$$

Denote by $\mathrm{sp}(T, L, x)$ the set of all (T, L)-spline interpolants of x, and let P (resp. $P^{\perp} := 1 - P$) be the orthoprojector onto L (resp. $L^{\perp}$). Then, by the first variational property of generalized inverses, the set $\mathrm{sp}(T, L, x)$ is explicitly represented (see Lemma 1.1) as

$$x - (TP^{\perp})^{\dagger}\, Tx + (\ker T \cap L^{\perp})$$

under the condition that

$$T(L^{\perp}) \text{ is closed.} \tag{1.2}$$

In [3], de Boor has already pointed out that $\mathrm{sp}(T, L, x)$ is nonempty and has a unique element for each $x \in H$ if and only if

$$\mathrm{incl}(\ker T, L^{\perp}) < 1, \tag{1.3}$$

where $\mathrm{incl}(M, N)$, the inclination between two linear subspaces M and N, is defined as the number

$$\sup\{|(m, n)| : m \in M, n \in N, \|m\| = \|n\| = 1\}.$$

Hence, by the representation of $\mathrm{sp}(T, L, x)$ we see [3] that (1.3) is equivalent to (1.2) and the following condition together:

$$\ker T \cap L^{\perp} = \{0\}. \tag{1.4}$$

Now, putting

$$S = 1 - (TP^{\perp})^{\dagger}T \qquad ((1.2) \text{ is assumed}),$$

we have the following basic result on spline interpolants:

LEMMA 1.1. *Let $T \in \mathrm{CR}(H, K)$ and let P be the orthoprojector onto a closed linear subspace L with* (1.2). *Then*

(1) *For each $x \in H$, an element $y \in H$ is in* $\mathrm{sp}(T, L, x)$ *if and only if $y = Sx + w$ for some $w \in \ker T \cap L^{\perp}$.*

(2) *The vector $y_0 = SPx$ is the* (*unique*) *element in* $\mathrm{sp}(T, L, x)$ *with minimum norm.*

(3) *An operator R on H* (*i.e., $R \in B(H, H)$*) *which maps each $x \in H$ to an element in* $\mathrm{sp}(T, L, x)$ *is representable as $R = S + W$ with some W on H satisfying* $\operatorname{ran} W \subset \ker T \cap L^{\perp}$.

Proof. Assertion (3) is easily obtained from (1). To see (1), let $y = x - P^{\perp}u \in x + L^{\perp}$, $u \in H$. Then $\|Ty\| = \|TP^{\perp}u - Tx\|$. Hence, by (1.1) $\|Ty\|$ is minimum if and only if $u = (TP^{\perp})^{\dagger}\, Tx + \{1 - (TP^{\perp})^{\dagger}(TP^{\perp})\}\, w$ with some $w \in H$. This is equivalent to

$$y = x - P^{\perp}[(TP^{\perp})^{\dagger}\, Tx + \{1 - (TP^{\perp})^{\dagger}(TP^{\perp})\}\, w] = Sx + w_1$$

for some $w_1 \in \ker T \cap L^{\perp}$. Assertion (2) could be obtained from [9, p. 228], but we give a direct proof for completeness; it suffices to show that for any $y \in \mathrm{sp}(T, L, x)$ $\|y\|^2 = \|y - SPx\|^2 + \|SPx\|^2$ or that $y - SPx$ and SPx are orthogonal. Since $SP^{\perp}x = \{1 - (TP^{\perp})^{\dagger}(TP^{\perp})\}\, P^{\perp}x \in \ker T \cap L^{\perp}$, we have, by (1), that $y - SPx = (y - Sx) + SP^{\perp}x \in \ker T \cap L^{\perp}$. On the other hand,

$$\begin{aligned} SPx &= Px - (TP^{\perp})^{\dagger}\, TPx \in L + \operatorname{ran}(TP^{\perp})^{\dagger} = L + \operatorname{ran}(TP^{\perp})^{*} \\ &= L + (\ker TP^{\perp})^{\perp} \subset (\ker T \cap L^{\perp})^{\perp}. \end{aligned}$$

Hereafter we shall call an operator R as in Lemma 1.1(3) a *(T, L)-spline operator.* Clearly, both S and SP are such operators, but, in addition, they are projectors (or idempotent operators). We shall call such projectors *(T, L)-spline projectors.*

As a key fact for our further discussions, we state a lemma on generalized inverses.

LEMMA 1.2 (cf. [10]). *Let $A \in \mathrm{CR}(H, K)$ and $B \in \mathrm{CR}(I, H)$* (*where I, H, and K are Hilbert spaces.*) *Then $A^{\dagger}ABB \in \mathrm{CR}(H) := \mathrm{CR}(H, H)$, and*

$$\|(AB)^{\dagger}\| \leqslant \|A^{\dagger}\|\, \|B^{\dagger}\|\, \|(A^{\dagger}ABB^{\dagger})^{\dagger}\|. \tag{1.5}$$

In particular, if A is invertible, then $\|(AB)^{\dagger}\| \leqslant \|A^{-1}\|\, \|B^{\dagger}\|$.

Proof. Let $C = A^{\dagger}ABB^{\dagger}$ and $D = B(AB)^{\dagger}A$. Then, by the Penrose identities for A, B, and AB, we have $CDC = C$. This identity implies that $C \in \mathrm{CR}(H)$. Inequality (1.5) is now obtained from the identities

$$\begin{aligned}(AB)^\dagger &= (AB)^\dagger AA^\dagger ABB^\dagger C^\dagger A^\dagger ABB^\dagger B(AB)^\dagger \\ &= (AB)^\dagger (AB) \cdot B^\dagger C^\dagger A^\dagger \cdot (AB)(AB)^\dagger.\end{aligned}$$

For the case when A is invertible, it suffices to note that C becomes an-orthoprojector and $C^\dagger = C$.

2. Convergence of Generalized Inverses

The convergence problem of generalized inverses is clearly identical to the perturbation problem of them. There are a number of results on this problem; see Nashed [12] or Stewart [14] which contain many aspects for perturbation theory of generalized inverses and contain some new results. In this section we shall, for our later use, show some necessary and sufficient conditions for the uniform or strong convergence of generalized inverses (which are seemed not to have appeared).

Let A_n $(n = 1, 2, \ldots)$ and A be operators in $B(H, K)$. We then write $A_n \to^{\mathrm{u}} A$ if the sequence $\{A_n\}$ converges to A uniformly, and $A_n \to^{\mathrm{s}} A$ if it converges to A strongly. When all A_n and A are in $\mathrm{CR}(H, K)$, under what condition is it true that $A_n^\dagger \to^{\mathrm{u}} A^\dagger$? When H and K are finite-dimensional, i.e., all operators are matrices, the convergence $A_n^\dagger \to^{\mathrm{u}} A^\dagger$ is guaranteed if (and only if) $\operatorname{rank} A_n = \operatorname{rank} A$ for all sufficiently large n [12, Theorem 3.5] or, equivalently, $A_n A_n^\dagger \to^{\mathrm{u}} AA^\dagger$ (cf. [14, Theorem 2.3]). We shall show that this is also true in the general case.

Lemma 2.1 [14, Theorems 3.2 and 3.3]. *Let $A, B \in \mathrm{CR}(H, K)$. Then*

$$\begin{aligned}B^\dagger - A^\dagger = -B^\dagger(B-A)A^\dagger &+ B^\dagger B^{*\dagger}(B^* - A^*)(AA^\dagger)^\perp \\ &+ (B^\dagger B)^\perp (B^* - A^*) A^{*\dagger} A^\dagger. \qquad (2.1)\end{aligned}$$

$$\|B^\dagger - A^\dagger\| \leqslant 3 \max\{\|B^\dagger\|^2, \|A^\dagger\|^2\} \|B - A\|. \qquad (2.2)$$

Lemma 2.2. *Let $A, B \in \mathrm{CR}(H, K)$, and let $\|B - A\| < \|A^\dagger\|^{-1}$, $\|BB^\dagger - AA^\dagger\| < 1$. Then $\|B^\dagger\| \leqslant 2\|A^\dagger\|(1 - \|A^\dagger\| \|B - A\|)^{-1}$.*

Proof. Write $P = AA^\dagger$ and $Q = BB^\dagger$. Then, since $\|P^\perp Q\|^2 = \|QP^\perp Q\| = \|Q(Q-P)Q\| < 1$, we see that $1 - P^\perp Q$ is invertible. We also see that $1 + A^\dagger(B - A)$ is invertible, because $\|A^\dagger(B - A)\| < 1$. Now, by the identity

$(1-P^{\perp}Q)B=A\{1+A^{\dagger}(B-A)\}$ or $B=(1-P^{\perp}Q)^{-1}A\{1+A^{\dagger}(B-A)\}$ and by Lemma 1.2, we have

$$\|B^{\dagger}\| \leqslant \|1-P^{\perp}Q\|\,\|A^{\dagger}\|\,\|\{1+A^{\dagger}(B-A)\}^{-1}\| \leqslant 2\,\|A^{\dagger}\|(1-\|A^{\dagger}\|\,\|B-A\|)^{-1}.$$

Remark. Under the stronger conditions $\|B-A\|<\|A^{\dagger}\|^{-1}$, $\|BB^{\dagger}-AA^{\dagger}\|<1$ and $\|B^{\dagger}B-A^{\dagger}A\|<1$, Wedin [15, Corollary 2 of Theorem 7.3] proved $\|B^{\dagger}\| \leqslant \|A^{\dagger}\|(1-\|A^{\dagger}(B-A)A^{\dagger}A\|)^{-1}$.

PROPOSITION 2.3. *Let* $\{A_n\}$ *be a sequence in* CR(H,K), *and let* $A_n \to^{\mathrm{u}} A \in$ CR(H,K). *Then the following conditions are equivalent*:

(1) $A_n^{\dagger} \to^{\mathrm{u}} A^{\dagger}$;

(2) $A_nA_n^{\dagger} \to^{\mathrm{u}} AA^{\dagger}$;

(2′) $A_n^{\dagger}A_n \to^{\mathrm{u}} A^{\dagger}A$.

Proof. That (1) ⇒ (2), (2′) is clear.

(2) ⇒ (1) Note, by (2.2), that $\|A_n^{\dagger}-A^{\dagger}\| \leqslant 3\max\{\|A_n^{\dagger}\|^2, \|A^{\dagger}\|^2\}\,\|A_n-A\|$. Hence we easily see that (1) is equivalent to

$$\sup_n \|A_n^{\dagger}\| < \infty. \tag{2.3}$$

To show (2.3), let n be sufficiently large. Then $\|A_nA_n^{\dagger}-AA^{\dagger}\|<1$ and $\|A_n-A\|<\|A^{\dagger}\|^{-1}$. Hence, by Lemma 2.2

$$\|A_n^{\dagger}\| \leqslant 2\,\|A^{\dagger}\|(1-\|A^{\dagger}\|\,\|A_n-A\|)^{-1}.$$

This implies (2.3).

(2′) ⇒ (1) In (2), replace A_n and A by their adjoints A_n^* and A^*, respectively.

We next show a simple (but equivalent) condition for the strong convergence of generalized inverses, which is to be compared with Proposition 2.3.

PROPOSITION 2.4. *Let* $\{A_n\}$ *be a sequence in* CR(H,K), *and let* $A_n \to^{\mathrm{s}} A \in$ CR(H,K). *Then the following conditions are equivalent*:

(1) $A_n^{\dagger} \to^{\mathrm{s}} A^{\dagger}$.

(2) $\sup_n \|A_n^{\dagger}\| < \infty$, $A_nA_n^{\dagger} \to^{\mathrm{s}} AA^{\dagger}$, *and* $A_n^{\dagger}A_n \to^{\mathrm{s}} A^{\dagger}A$.

Proof. If we assume (1), then the inequality in (2) is obtained from the Banach–Steinhaus theorem, and the other assertions in (2) are easily seen by the uniform boundedness of $\{A_n\}$ and $\{A_n^\dagger\}$. For the converse (2) $\Rightarrow$ (1), it suffices to note

$$A_n^\dagger - A^\dagger = (A_n^\dagger A_n - A^\dagger A) A^\dagger - A_n^\dagger (A_n - A) A^\dagger + A_n^\dagger (A_n A_n^\dagger - AA^\dagger).$$

Remarks. (1) In contrast to the case of uniform convergence, we cannot deduce the inequality $\sup_n \|A_n^\dagger\| < \infty$ from $A_n A_n^\dagger \to^{s} AA^\dagger$ (or $A_n^\dagger A_n \to^{s} A^\dagger A$). For example, let

$$A_n = \operatorname{diag}\{\overbrace{1,\ldots,1}^{n}, 1/n, 1/n,\ldots\} \qquad \text{on} \quad H := \ell^2.$$

(2) An operator $A^\phi \in B(K, H)$ is called an outer inverse of $A \in B(H, K)$ if $A^\phi A A^\phi = A^\phi$. Concerning the convergence of such general generalized inverses, Anselone and Nashed [1] proved that if $A_n \to^{u} A$ (resp. $A_n \to^{s} A$) and, for each n, A_n^ϕ is an outer inverse of A_n with $\operatorname{ran} A_n^\phi \supset \operatorname{ran} A^\phi$, $\ker A_n^\phi \supset \ker A^\phi$, and $\sup_n \|A_n^\phi\| < \infty$, then $A_n^\phi \to^{u} A^\phi$ (resp. $A_n^\phi \to^{s} A^\phi$).

The following result is on the relation between the strong convergence of $\{A_n^\dagger\}$ and $\{A_n^*\}$; we do not assume the convergence of $\{A_n\}$ itself, but add some weaker conditions:

PROPOSITION 2.5. *Let $\{A_n\}$ be a uniformly bounded sequence in* $\mathrm{CR}(H, K)$, *and let $A \in \mathrm{CR}(H, K)$. Then the following conditions are equivalent*:

(1) $A_n^\dagger \to^{s} A^\dagger$ *and* $A_n^{\dagger *} A^* \to^{s} AA^\dagger$ $(= A^{\dagger *} A^*)$.

(2) $\sup_n \|A_n^\dagger\| < \infty$, $A_n^* \to^{s} A^*$, *and* $A_n A^\dagger \to^{s} AA^\dagger$.

Proof. (1) $\Rightarrow$ (2) The inequality in (2) is clear (by the Banach–Steinhaus theorem). To see the convergence of $\{A_n^*\}$, replace B and A in (2.1) by $A_n^{\dagger *}$ and $A^{\dagger *}$, respectively. Then, using the identity $(C^{\dagger *})^\dagger = C^*$ $(C \in \mathrm{CR}(H, K))$, we have

$$\begin{aligned} A_n^* - A^* &= (A_n^{\dagger *})^\dagger - (A^{\dagger *})^\dagger \\ &= -A_n^*(A_n^{\dagger *} A^* - A^{\dagger *} A^*) + A_n^* A_n (A_n^\dagger - A^\dagger)(AA^\dagger)^\perp \\ &\quad + (A_n^\dagger A_n)^\perp (A_n^\dagger - A^\dagger) AA^*. \end{aligned}$$

Hence, since $\{A_n^\dagger\}$ and $\{A_n\}$ are uniformly bounded, we have $A_n^* \to^{s} A^*$. For the (strong) convergence of $\{A_n A^\dagger\}$, we have, for any $x \in H$,

$$\lim_{n\to\infty} A_n A^\dagger x = \lim_{n\to\infty} A_n A_n^\dagger x = \lim_{n\to\infty} A_n^{\dagger *} A_n^* x = \lim_{n\to\infty} A_n^{\dagger *} A^* x = AA^\dagger x.$$

(2) ⇒ (1) Again, we use (2.1). Since

$$A_n^\dagger - A^\dagger = -A_n^\dagger(A_nA^\dagger - AA^\dagger) + A_n^\dagger A_n^{*\dagger}(A_n^* - A^*)(AA^\dagger)^\perp \\ + (A_n^\dagger A_n)^\perp(A_n^* - A^*)(A^{*\dagger}A^\dagger),$$

and since $\{A_n^\dagger\}$ is uniformly bounded, we have $A_n^\dagger \to^{\mathrm{s}} A^\dagger$. To see the convergence of $\{A_n^{\dagger *}A^*\}$, let $x \in H$. Then

$$\lim_{n\to\infty} A_n^{\dagger *}A^*x = \lim_{n\to\infty} A_n^{\dagger *}A_n^*x = \lim_{n\to\infty} A_nA_n^\dagger x = \lim_{n\to\infty} A_nA^\dagger x = AA^\dagger x.$$

Remark. If we replace strong convergence by uniform convergence in the above proposition, then the assumptions on the convergence of $\{A_n^{\dagger *}A^*\}$ and $\{A_nA^\dagger\}$ will be redundant, and the proposition will say that $A_n^\dagger \to^{\mathrm{u}} A^\dagger$ if and only if $A_n \to^{\mathrm{u}} A$ and $\sup_n \|A_n^\dagger\| < \infty$, which was shown in the proof of Proposition 2.3, (2) ⇒ (1).

Putting $A = 0$ $(= A^\dagger)$ in Proposition 2.5, we have:

COROLLARY 2.6. *Let $\{A_n\}$ be a uniformly bounded sequence in* $\mathrm{CR}(H, K)$. *Then $A_n^\dagger \to^{\mathrm{s}} 0$ if and only if $\sup_n \|A_n^\dagger\| < \infty$ and $A_n^* \to^{\mathrm{s}} 0$.*

3. CONVERGENCE OF SPLINE PROJECTORS

Recall that for $T \in \mathrm{CR}(H, K)$ and a closed linear subspace L in H satisfying condition (1.2), i.e., that $T(L^\perp)$ is closed, the (T, L)-spline projector S is defined by $S = 1 - (TP^\perp)^\dagger T$, where P is the orthoprojector onto L. Let $\{L_n\}$ be a sequence of closed linear subspaces in H satisfying (1.2), and let $\{P_n\}$ and $\{S_n\}$ $(S_n = 1 - (TP_n^\perp)^\dagger T)$ be the corresponding orthoprojectors and spline projectors, respectively. Then, for the strong convergence of $\{S_n\}$ we have a refinement of a result due to de Boor [3, Theorem 2].

THEOREM 3.1. *Let $\{S_n\}$ be a sequence of spline projectors defined as above. Put $Q = T^\dagger T$. Then the following conditions are equivalent*:

(1) $S_n \to^{\mathrm{s}} 1$.

(2) $(TP_n^\perp)^\dagger \to^{\mathrm{s}} 0$.

(3) $\sup_n \|(TP_n^\perp)^\dagger\| < \infty$ *and* $P_n^\perp T^* \to^{\mathrm{s}} 0$.

(4) $\sup_n \|(QP_n^\perp)^\dagger\| < \infty$ *and* $P_n^\perp Q \to^{\mathrm{s}} 0$.

Proof. (1) $\Leftrightarrow$ (2) Since $(TP_n^\perp)^\dagger = (TP_n^\perp)^\dagger TT^\dagger = (TP_n^\perp)^\dagger T \cdot T^\dagger$, we have $(TP_n^\perp)^\dagger \to^s 0$ if and only if $(TP_n^\perp)^\dagger T \to^s 0$.

(2) $\Leftrightarrow$ (3) This is true by Corollary 2.6.

(3) $\Leftrightarrow$ (4) By (1.5) we have

$$\|(TP_n^\perp)^\dagger\| \leqslant \|T^\dagger\| \, \|P_n^\perp\| \, \|(T^\dagger TP_n^\perp)^\dagger\| \leqslant \|T^\dagger\| \, \|(QP_n^\perp)^\dagger\|$$

and

$$\begin{aligned}\|(QP_n^\perp)^\dagger\| &= \|(T^\dagger \cdot TP_n^\perp)^\dagger\| \leqslant \|T\| \, \|(TP_n^\perp)^\dagger\| \, \|\{TT^\dagger(TP_n^\perp)(TP_n^\perp)^\dagger\}\| \\ &= \|T\| \, \|(TP_n^\perp)^\dagger\| \, \|(TP_n^\perp)(TP_n^\perp)^\dagger\| \leqslant \|T\| \, \|(TP_n^\perp)^\dagger\|.\end{aligned}$$

Hence the uniform boundedness of $\{(TP_n^\perp)^\dagger\}$ and $\{(QP_n^\perp)^\dagger\}$ are equivalent. For the equivalence of the strong convergence, it suffices to note that $Q = T^* \cdot T^{\dagger *}$ and $T^* = Q \cdot T^*$.

Remarks. (1) For the product of two orthoprojectors Q and R on H with $QR \in \mathrm{CR}(H)$ we know [10] that

$$\|(QR)^\dagger\|^{-2} + \|Q^\perp(Q^\perp \wedge R)^\perp R\|^2 = 1, \tag{3.1}$$

where $Q^\perp \wedge R$ is the orthoprojector onto $\operatorname{ran} Q^\perp \cap \operatorname{ran} R$. Hence the inequality $\sup_n \|(QP_n^\perp)^\dagger\| < \infty$ in the theorem is equivalent to

$$\sup_n \|Q^\perp(Q^\perp \wedge P_n^\perp)^\perp P_n^\perp\| < 1.$$

We easily see that this inequality means nothing but

$$\sup_n \operatorname{incl}(\ker T \cap (\ker T \cap L_n^\perp)^\perp, L_n^\perp) < 1.$$

(2) Define $\underline{\lim}\, L_n = \{x : \operatorname{dist}(x, L_n) \to 0\}$. Then it is easy to see that the condition $P_n^\perp Q \to^s 0$ in the theorem is equivalent to $(\ker T)^\perp \subset \underline{\lim}\, L_n$ (cf. [3]).

By Lemma 1.1(3), all (T, L_n)-spline operators R_n are represented as $R_n = S_n + W_n$ with some W_n satisfying $\operatorname{ran} W_n \subset \ker T \cap L_n^\perp$. For the convergence of such operators we have

PROPOSITION 3.2. *Let $\{R_n\}$ be a sequence of spline operators as above. Then, $R_n \to^s 1$ if and only if $S_n \to^s 1$ and $W_n \to^s 0$.*

Proof. It suffices to show that for any $x \in H$, $\|(R_n - 1)x\|^2 =$

$\|(S_n - 1)x\|^2 + \|W_n x\|^2$ or that $(S_n - 1)x$ and $W_n x$ are orthogonal. Since $W_n x \in \ker T \cap L_n^\perp$, and since

$$\begin{aligned}(S_n - 1)x &= (TP_n^\perp)^\dagger\, Tx \in \operatorname{ran}(TP_n^\perp)^\dagger = \operatorname{ran}(TP_n^\perp)^* \\ &= (\ker TP_n^\perp)^\perp \subset (\ker T \cap L_n^\perp)^\perp,\end{aligned}$$

we obtain the desired relation.

In case $\dim \ker T < \infty$, the following result holds; it was shown by Shekhtman [13, Theorem 1] and de Boor [3, Theorem 1]. We give a different proof, using generalized inverses.

THEOREM 3.3. *Let $\{S_n\}$ be a sequence of spline projectors defined as before. If* $\dim \ker T < \infty$, *then $P_n \to^{s} 1$ implies $S_n \to^{s} 1$.*

Proof. Write $Q = T^\dagger T$. Then clearly $P_n^\perp Q \to^{s} 0$. To see $S_n \to^{s} 1$, it suffices, by Theorem 3.1, to show that $\{(QP_n^\perp)^\dagger\}$ is uniformly bounded. Since $Q^\perp$ has finite rank, we easily see $P_n^\perp Q^\perp \to^{u} 0$. The uniform boundedness of $\{(QP_n^\perp)^\dagger\}$ is obtained from (putting $R = P_n^\perp$ in (3.1) or) the identity

$$\|(QP_n^\perp)^\dagger\| \leqslant \|(1 - Q^\perp P_n^\perp)^{-1}\| \quad (\to 1), \tag{3.2}$$

which is seen by the identity $(1 - Q^\perp P_n^\perp)(QP_n^\perp)^\dagger = (QP_n^\perp)(QP_n^\perp)^\dagger$ or $(QP_n^\perp)^\dagger = (1 - Q^\perp P_n^\perp)^{-1} \cdot (QP_n^\perp)(QP_n^\perp)^\dagger$.

The following result is a modification of [3, Proposition 2]; we could give a proof similar to the one in [3], but instead adopt the generalized inverse method again:

PROPOSITION 3.4. *Let $S_n \to^{s} 1$ (S_n is defined as before), and let* $\dim \ker T < \infty$. *Then $P_n \to^{s} 1$ if and only if there exists a sequence $\{R_n\}$ of projectors and a projector R on H such that*

$$R_n \xrightarrow{\mathrm{u}} R, \qquad R_n R_n^\dagger \xrightarrow{\mathrm{u}} (T^\dagger T)^\perp, \qquad \text{and} \qquad P_n R_n^\dagger R_n = R_n^\dagger R_n. \tag{3.3}$$

Proof. Write $Q = T^\dagger T$. If $P_n \to^{s} 1$, then since rank $Q^\perp$ is finite we see $P_n Q^\perp \to^{u} Q^\perp$ and $P_n^\perp Q^\perp \to^{u} 0$. Put $R_n = (P_n Q^\perp)^\dagger$. Then we see that each R_n is a projector and the sequence $\{R_n\}$ is uniformly bounded, say, by (3.2) (exchange P_n and Q). Hence $R_n \to^{u} (Q^\perp)^\dagger = Q^\perp$ (cf. proof of Proposition 2.3). Putting $R = Q^\perp$, we at once obtain all the conditions in (3.3). Conversely, assume that R_n and R are projectors satisfying (3.3). Then, taking the limits of $R_n = R_n R_n^\dagger \cdot R_n$ and $R_n R_n^\dagger = R_n \cdot R_n R_n^\dagger$, we see that $\operatorname{ran} R = \operatorname{ran} Q^\perp$ or $RR^\dagger = Q^\perp$. Hence $R_n R_n^\dagger \to^{u} RR^\dagger$, so that $R_n^\dagger R_n \to^{u} R^\dagger R$, say, by Proposition 2.3. Hence we have $P_n R^\dagger R \to^{s} R^\dagger R$. Since $S_n \to^{s} 1$, we also have $P_n^\perp Q \to^{s} 0$ or $P_n Q \to^{s} Q$ by Theorem 3.1. Hence all we have to do is to show that $U := R^\dagger R + Q$ is invertible or equivalently strictly positive (i.e., $(Ux, x) > 0$ for any $x \neq 0$). Note that $\ker T = \operatorname{ran} R$ and R is a projector. Hence $(\ker T)^\perp = (\operatorname{ran} R)^\perp = (\ker(1-R))^\perp$, that is, $Q = (1-R)^\dagger(1-R)$. Since $A^*A \leqslant \|A\|^2 A^\dagger A$ (i.e., $(A^*Ax, x) \leqslant \|A\|^2(A^\dagger Ax, x)$) for $A \in \mathrm{CR}(H)$, we have

$$U = R^\dagger R + (1-R)^\dagger(1-R) \geqslant \|R\|^{-2} R^*R + \|1-R\|^{-2}(1-R)^*(1-R)$$
$$\geqslant m\{R^*R + (1-R)^*(1-R)\} \geqslant m \cdot \tfrac{1}{2}\{R + (1-R)\}^*\{R + (1-R)\} = \tfrac{1}{2}m,$$

where $m = \min\{\|R\|^{-2}, \|1-R\|^{-2}\}$. This completes the proof.

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